Multisoliton collisions in nearly integrable systems

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(Received 1 April 1996; revised manuscript received 28 May 1996)

We describe basic features of multisoliton collisions in nearly integrable systems taking a perturbed nonlinear Schrödinger equation as an example. Collision of two solitons is shown to become inelastic only due to radiation losses, so that the change of the soliton parameters is small ($\sim\epsilon^2$, where ϵ is the perturbation amplitude). For three-soliton collisions we demonstrate, by using a simplectic numerical integration, the existence of a nontrivial nonradiative energy exchange between the colliding solitons already in the first order in ϵ. [S1063-651X(96)51909-4]

PACS number(s): 03.40.Kf

One of the most remarkable properties of *solitons*, localized nonlinear waves which propagate without change of their shapes and velocities, is their elastic collisions, as was discovered first for the Korteweg–de Vries equation [1]. As has been well understood for a variety of integrable models (see, e.g., [2]), interaction of solitons results only in a shift of their phases, the shift due to the collision with several solitons being equal to the sum of partial shifts resulting from separate collisions with each soliton. This property is commonly referred to as the absence of *multisoliton* (or ''many-particle'') effects in integrable models. Because integrable models appear as a limit of more general equations, they describe the physical systems only in a certain asymptotic limit, and very often one needs to know effects produced by nonintegrability of the nonlinear equations. Then the natural question arises: *What is the result of multisoliton collisions in the physical models described by nearly integrable equations?* It is commonly believed that the main difference is due to radiation emitted by the interacting solitons [3]. However, in the present paper, undertaking extended numerical simulations based on a simplectic integration scheme, we demonstrate the existence of nontrivial effects in multisoliton collisions which do not involve radiation and exist at any value of the perturbation parameter ϵ. These effects are the energy exchange between the colliding solitons and excitation of internal soliton modes which, as we believe, are the major effects which distinguish multisoliton collisions in integrable and nonintegrable models.

To demonstrate the main features of multisoliton collisions in nearly integrable systems, we consider the nonlinear Schrödinger (NLS) equation with a small quintic nonlinearity,

$$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+2|u|^2u=\epsilon|u|^4u, \tag{1}$$

where u is the complex wave envelope, and t and x are time and coordinate, respectively. The parameter ϵ in Eq. (1) is the perturbation amplitude which is assumed to be small. Equation (1) can appear in different physical problems; in particular, it describes evolution of the electric field in an optical waveguide with the intensity-dependent refractive index $n_{\rm nl}(I\equiv|u|^2)$, which slightly deviates from the Kerr dependence. Nonideality of the nonlinear optical response is known for semiconductor waveguides (e.g., $Al_xGa_{1-x}As$ or $CdS_{1-x}Se_x$) or nonlinear polymers (e.g., *p*-toluene sulfonate). In the latter case, the nonlinear refractive index can be modeled by cubic-quintic nonlinearity [4], $n_{\rm nl}(I)=n_2I+n_3I^2$, which leads directly to the model (1).

In the absence of perturbation ($\epsilon=0$), the NLS equation (1) is known to be exactly integrable [5] and it supports propagation of an envelope soliton with the amplitude a and velocity V. Additionally, Eq. (1) possesses an infinite number of integrals of motion [2,5]. Three elementary integrals are the norm, N, field momentum, P, and energy, E (see definitions, e.g., in Ref. [6]). Calculated for the NLS soliton, these values are [6], p. 778)

$$N_s=2a,\quad P_s=aV,\quad E_s=\tfrac{1}{2}aV^2-\tfrac{2}{3}a^3. \tag{2}$$

Unlike the higher (nonelementary) integrals of motion, these three basic invariants remain conserved for the perturbed NLS equation (1) as well.

The effect of the conservative perturbation in Eq. (1) on the soliton is trivial (see Ref. [6]). This is consistent with the fact that a perturbed equation has an exact solitary wave solution which is *a slightly modified* NLS soliton. However, interaction of these solitary waves differs drastically from the interaction of solitons of the integrable NLS equation. In general, even for the case of two solitons such interactions

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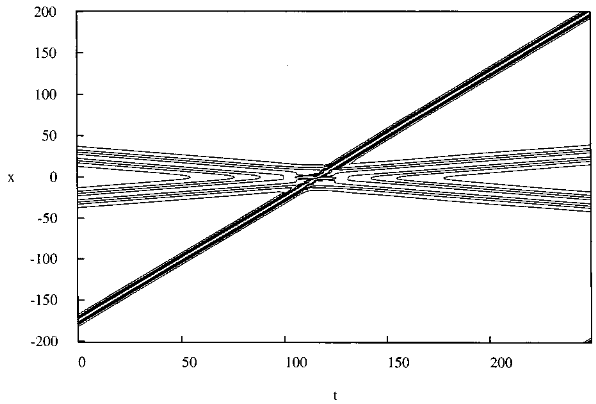

FIG. 1. The triple-soliton collision. The fast soliton with the amplitude a_f and velocity V_f collides with two slow solitons of equal amplitudes a_{sl} propagating towards each other with velocities $\pm V_{sl}$, so that at the collision moment all the solitons are strongly overlapping.

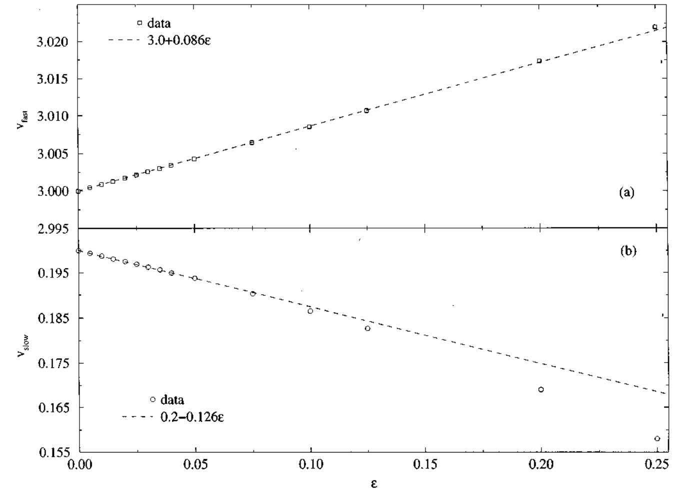

FIG. 2. Final soliton velocities after triple soliton collision for different values of ϵ: (a) the fast soliton with the initial velocity $V_f=3.0$; (b) one of the slow solitons with the initial velocity $V_{sl}=0.2$. Open squares and circles are numerical data, dashed lines are linear fitted functions.

are better analyzed numerically. To investigate multisoliton collisions, we take Eq. (1) as a typical example and integrate it using the fourth-order symplectic integrator as described in Ref. [7]. The symplectic numerical method we used is much better than traditional numerical schemes, as during the integration time it allows one to preserve the norm with the relative accuracy 10^{-11}, and the energy with the accuracy 10^{-6}. The grid spacing we used is $dx=0.1$ with the total length $L=[-800,800]$ and the time step $dt=0.005$. We are interested in collisions of three solitons but, for comparison, we have considered the case of two solitons as well. In the case of three solitons, one of them (we call it ''fast''), with a relatively large velocity, was taken as an exact solution of Eq. (1) to avoid initial oscillation of its amplitude. Two other (''slow'') solitons were modeled by an exact two-soliton solution of the unperturbed NLS equation to avoid radiation due to strong initial overlapping. The solitons were put on the grid at the positions $x_f^{(0)}=-650$ and $\pm x_{sl}^{(0)}=\pm 45$ with the initial amplitudes $a_f=1/\sqrt{2}$, $a_{sl}=0.35$ and the velocities $V_f=3.0$ and $\pm V_{sl}=\pm 0.2$, selected in such a way that all three solitons collide when the two slow ones are strongly overlapping (see Fig. 1).

First, we consider two-soliton collisions. Simulations for two solitons were performed for the same initial data as mentioned above but with only one slow soliton (with negative velocity) instead of two. For $\epsilon<0.1$, the change of the soliton velocities after the interaction was found to be so small that we were not able to measure it with a sufficient resolution, so that it is expected to be on the order of a numerical error. To understand this result, we note that for two solitons inelastic effects in the collision may appear only due to emission of radiation. Emitted energy E_{rad} has been calculated analytically (Ref. [6], p. 863) in the limit of two symmetric solitons with the equal amplitudes $a_1=a_2=a$ and the velocities $V_{1,2}=\pm V$, provided that $V \gg a$. The result is $E_{rad}=C\epsilon^2 a^7\{1+F(V/a)\}$, where C is a numerical constant. Analogously, we can find the radiation-induced change of the norm [6], $N_{rad}=(4/V^2)E_{rad}$. Now, we can use these results to describe a change of the soliton parameters after the interaction. Indeed, because of the symmetry, the radiation does not change the total momentum of two solitons. Now, taking into account the conservation of the total norm and energy, we can write $N_s=N_s'+N_{rad}$ and $E_s=E_s'+E_{rad}$, where the values with the subscripts ''s'' are defined in Eq. (2); the left-hand sides pertain to the initial solitons before the collisions, whereas the right-hand sides take into account a change due to the radiation emitted. These two *balance equations* for the conserved quantities allow us to find the change of the solitons' amplitudes Δa and velocities ΔV,

$$\Delta a=\frac{1}{V^2}E_{rad}, \quad \Delta V=\frac{2a}{V^3}E_{rad}, \tag{3}$$

where E_{rad} is defined above. According to Eqs. (3), the change of the soliton parameters is proportional to ϵ^2. Another important result is the inversely proportional dependence on V^2: For the solitons colliding with large velocities V, the interaction time is small and therefore the change of the soliton parameters is smaller too. For two nonequal solitons this result will be attenuated by smaller time of the soliton overlapping, in agreement with our numerical simulations.

Unlike the case of two solitons, for the collision of three solitons we obtain nontrivial effects already in the first order in ϵ. Figures 2(a) and 2(b) show the change of the soliton velocities after the collision for different values of the perturbation amplitude ϵ. It is clear that the change is *linear* in ϵ. After collision we also observe small oscillations of the soliton amplitudes, the effect more visible for larger ϵ; see Figs. 3(a) and 3(b). Both these effects differ drastically in the collision of two and three solitons.

The first effect indicates the existence of a nontrivial energy exchange in nearly integrable models already in the first order of the perturbation amplitude ϵ. As a matter of fact, this effect was first mentioned in Ref. [8] for the very special relation between the soliton parameters. To find analytical results in that approximation, we consider a collision between a fast soliton with the amplitude a_f and the velocity V_f, and a symmetric pair of two ''slow'' solitons with equal

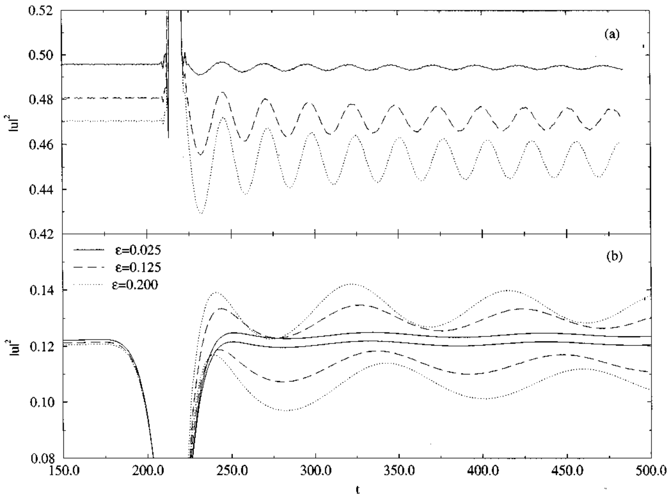

FIG. 3. Examples of the variation of the soliton amplitudes after the interaction for $\epsilon=0.025,0.125$, and 0.2 for (a) the fast soliton, and (b) two slow solitons.

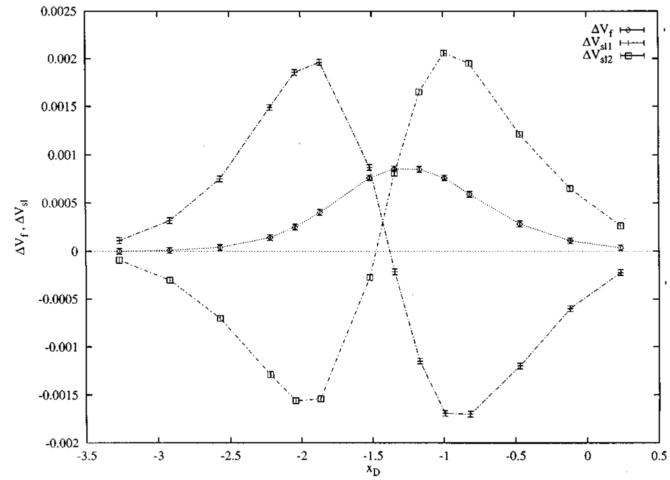

FIG. 4. Change of the soliton velocities vs the extrapolated initial distance between the slow solitons, x_D. Shown are the differences of the velocities for the fast soliton, ΔV_f (dotted curve), and for the slow solitons, $\Delta V_{\rm sl}$ (dash-dotted curves). Three types of the marks indicate the data obtained by direct numerical simulations at $\epsilon=0.01$.

amplitudes $a_{\rm sl}$ and opposite velocities $V_{\rm sl}$. Following Ref. [8], calculations can be performed only under the following assumptions about the soliton parameters: $V_f \gg V_{\rm sl} \gg a_{\rm sl}$. This assumption allows one to present the three-soliton solution in the form of the sum, $u=u_f(V_f)+u_{\rm sl}(V_{\rm sl})+u_{\rm sl}(-V_{\rm sl})$, and calculate the change of the soliton parameters by means of the soliton perturbation theory based on the inverse interaction transform using the one-soliton Jost functions (see details in Refs. [6,8]). In the case when the initial amplitudes of the slow solitons are equal, the soliton parameters after the collision are given by the following expressions: $V_f' = V_f + \Delta V_f$ and $\pm V_{\rm sl}' = \pm V_{\rm sl} \mp \Delta V_{\rm sl}$, where

$$\Delta V_f = -192\epsilon \frac{V_{\rm sl} a_{\rm sl}^4}{V_f^2} G(\delta), \quad \Delta V_{\rm sl} = 96\epsilon \frac{a_f a_{\rm sl}^3}{V_f} G(\delta); \quad (4)$$

the parameter $\delta = a_{\rm sl}(x_{\rm sl2}^{(0)} - x_{\rm sl1}^{(0)})$ characterizes the separation between the slow solitons at the moment of their collision with the third (fast) soliton, and the odd function $G(\delta)$ is

$$G(\delta) = \frac{1}{\sinh^2\delta}\left[\frac{3(\delta - \tanh\delta)}{\tanh^2\delta} - \delta\right], \quad (5)$$

which vanishes at $\delta \to 0$ and $\delta \to \infty$. At the same order of the perturbation theory, there is no change of the soliton amplitudes.

It is easy to verify that the results (4) are consistent with the conservation laws. Indeed, we find that $\Delta N = \Sigma_j \Delta a_j = 0$ because the soliton amplitudes do not change after the collision; the total momentum is conserved up to the order V_f^{-2} due to the symmetry of the problem, whereas for the energy we obtain

$$\Delta E = (2a_{\rm sl}V_{\rm sl}\Delta V_{\rm sl} + a_f V_f \Delta V_f). \quad (6)$$

Substituting the values from Eq. (4), we verify that $\Delta E = 0$.

Generally speaking, the analytical results (4) are valid in a relatively narrow region of the soliton parameters. However, they predict a nontrivial dependence on the relative distance between the slow solitons, which can also be found in a more general case. To analyze the dependence of the energy exchange between the solitons on the soliton separation, we fix the value of the perturbation amplitude to be $\epsilon=0.01$ and vary the initial separation between the slow solitons. This resulted in a variation of the averaged distance between the slow solitons at the moment of interaction, i.e., effectively δ in Eq. (5). To measure the separation distance, we consider simultaneously the collision between the slow solitons under the same conditions but *without* the third soliton. The values of x_D are then measured at the moment of time when the fast soliton is passing the point $x=0$. The final velocities were measured with the help of the linear regression analysis using the program XMgr $v3.01$. It is clear that another way to define x_D will give only a shift of all the values by a constant. The numerical results are summarized in Fig. 4, where we show the relative changes of the soliton velocities, $\Delta V_{\rm f}$ and $\Delta V_{\rm sl}$, as functions of the extrapolated distance between the slow solitons. The results obtained are qualitatively similar to those predicted by theory. Indeed, the change of the velocities of slow solitons is due to the energy exchange during the collision, the effect which strongly depends on the separation between the colliding solitons at the moment of collision, and it vanishes for larger separations. Additionally, this energy exchange vanishes when the centers of the slow solitons almost coincide.

However, the change of the velocity of the fast soliton as observed in numerical simulations differs from what is predicted by theory. More detailed analysis indicates that the energy exchange is more complicated and it involves, at least in this region of the soliton parameters, excitation of an internal mode of the fast soliton. As a matter of fact, this internal mode appears as a nontrivial *localized* eigenmode of the linear problem associated with the soliton of the perturbed NLS equation (1) and it always exists for $\epsilon > 0$. From the physical point of view, this mode describes long-lived oscillations of the soliton amplitude. This kind of mode is known for other types of the soliton-bearing models but, to the best of our knowledge, it was not investigated in detail for the envelope solitons. Here we discover, to our surprise, an important role of the soliton internal mode which participates in the energy exchange between the solitons during three-soliton collisions. This effect calls for further investigation, and we can expect, by analogy with the resonant kink-kink and kink-impurity interactions [9], the existence of similar resonances for envelope solitons.

In conclusion, we have presented results of numerical simulations of the three-soliton collisions in the weakly perturbed NLS equation, which demonstrate nontrivial inelastic effects. We have found that, unlike the case of two colliding solitons, the collision of three solitons is accompanied by a radiationless energy exchange between them and excitation of internal modes of the colliding solitons. Both these effects lead to a change of the soliton velocities at the first order of the perturbation amplitude ϵ. The effect depends nontrivially on the relative distance between the solitons, and it vanishes in the limit of strongly separated solitons.

Yuri Kivshar thanks Mark Ablowitz, Dmitry Pelinovsky, and Allan Snyder for useful remarks. Helge Frauenkron is supported by DFG within the Graduiertenkolleg ''Feldtheoretische und Numerische Methoden in der Elementarteilchen-und Statistischen Physik.''

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